

Wyman's solution, self-similarity and critical behaviour.

G. Oliveira-Neto* and F. I. Takakura†

Departamento de Física, Instituto de Ciencias Exatas, Universidade Federal de Juiz de Fora,

CEP 36036-330, Juiz de Fora, Minas Gerais, Brazil.

(February 7, 2008)

Abstract

We show that the Wyman's solution may be obtained from the four-dimensional Einstein's equations for a spherically symmetric, minimally coupled, massless scalar field by using the continuous self-similarity of those equations. The Wyman's solution depends on two parameters, the mass M and the scalar charge Σ . If one fixes M to a positive value, say M_0 , and let Σ^2 take values along the real line we show that this solution exhibits critical behaviour. For $\Sigma^2 > 0$ the space-times have eternal naked singularities, for $\Sigma^2 = 0$ one has a Schwarzschild black hole of mass M_0 and finally for $-M_0^2 \leq \Sigma^2 < 0$ one has eternal bouncing solutions.

04.20.Dw,04.40.Nr,04.70.Bw

Typeset using REVTeX

*Email: gilneto@fisica.ufjf.br

†Email: takakura@fisica.ufjf.br

I. INTRODUCTION.

The four dimensional space-time generated by a minimally coupled, spherically symmetric, static, massless scalar field has been studied by many authors [1–5]. The general solution was found by M. Wyman in [2]. From a particular case of the general solution, M. D. Roberts showed how to construct a time dependent solution [3]. This solution has an important physical interest because it may represent the gravitational collapse of a scalar field. Later, P. Brady and independently Y. Oshiro et al. [6] showed that the Roberts' solution could be derived from the appropriated, time-dependent, Einstein-scalar equations by using a continuous self-similarity. They also showed that this solution exhibits a critical behaviour qualitatively identical to the one found numerically by M. W. Choptuik [7], studying the same system of equations.

Using a discrete self-similarity of the system, Choptuik explicitly showed that, the collapse results in two families of solutions depending on the value of a certain parameter p . p characterizes the scalar charge. In the first family, when $p < p_c$, the scalar field collapses up to a certain surface and then disperses. In the other family, when $p > p_c$, the scalar field collapses to form a black hole. The critical value p_c separates the two end states of the collapse. The critical solution ($p = p_c$) represents a naked singularity. Therefore, since it is given by a single value of the parameter, it has zero measure in the parameter space. In fact, the above results confirmed early studies of D. Christodoulou who pioneered analytical studies of that model [8].

The Wyman's solution is not usually thought to be of great importance for the issue of gravitational collapse because it is static and the naked singularities derived from it are unstable against spherically symmetric linear perturbations of the system [4,5]. On the other hand, as we saw above, from a particular case of this solution one may derive the Roberts' one which is of great importance for the issue of gravitational collapse. Also, it was shown that there are nakedly singular solutions to the static, massive scalar field equations which are stable against spherically symmetric linear perturbations [5]. Therefore, we think it is of

great importance to gather as much information as we can about the Wyman's solution for they may be helpful for a better understand of the scalar field collapse.

In the present work, we would like to show that the Wyman's solution may be obtained from the four-dimensional Einstein's equations for a spherically symmetric, minimally coupled, massless scalar field by using the continuous self-similarity of those equations. The way by which we shall re-write the Einstein-scalar equations in terms of a single variable is different from the one used in the derivation of the Roberts' solution in [6]. We use a coordinate system with two null coordinates (u, v) and the usual angular coordinates. Due to the spherical symmetry all the functions appearing in the field equations must depend only on u and v . Our system is characterized by three functions: σ and r , present in the metric and the scalar field Φ . Using the continuous self-similarity of the equations we re-write all functions in terms of a single variable $z \equiv v/u$, in the following way: $\exp(2\sigma) \rightarrow -f_{,u} f_{,v}$ and $f(u, v) \rightarrow f(z)$, where $f(u, v)$ is an auxiliary function; $r(u, v) \rightarrow r(z)$; and $\Phi(u, v) \rightarrow \Phi(z)$. Therefore, the difference between the way the authors in [6] re-write the Einstein-scalar equations in terms of a single variable and ours, is due to the σ and r transformations. We believe that the above described way, may be systematically used in other self-similar system of equations. After that, we find the differential equations for the variables r , σ , Φ and show that the Wyman's solution, in the appropriate coordinates, satisfies them.

The Wyman's solution depends on two parameters, the mass M and the scalar charge Σ . As we have seen above, Choptuik found a critical behaviour in the solutions to the Einstein-scalar equations in terms of the parameter describing the scalar charge. Therefore, one may try to find critical behaviour in Wyman's solution by fixing M and letting Σ varies. We would like to show that, if one fixes M to a positive value, say M_0 , and let Σ^2 take values along the real line, this solution exhibits critical behaviour. For $\Sigma^2 > 0$ the space-times have eternal naked singularities, for $\Sigma^2 = 0$ one has a Schwarzschild black hole of mass M_0 and finally for $-M_0^2 \leq \Sigma^2 < 0$ one has eternal bouncing solutions. Here, we can see that the critical solution is a Schwarzschild black hole which interpolates between an infinity number of eternal naked singularities and bouncing space-times. Although, this behaviour is very

different from the one discovered by Choptuik, one may have other types of critical solutions in the Einstein-scalar system, as demonstrated by P. Brady in [11].

In the next section, Section II, we show that the Wyman's solution may be obtained from the four-dimensional Einstein's equations for a spherically symmetric, minimally coupled, massless scalar field by using the continuous self-similarity of those equations.

In Section III, we show that the Wyman's solution exhibits critical behaviour. In particular, we numerically solve, with the appropriate boundary conditions, the equation for the radial function (r) in terms of the radial coordinate (R) and show how this function behaves for each type of solution: eternal naked singularities, eternal bouncing solutions and the Schwarzschild black hole.

Finally, in Section IV, we summarize the main points and results of the paper.

II. WYMAN'S SOLUTION AND CONTINUOUS SELF-SIMILARITY.

We shall start by writing down the ansatz for the space-time metric. As we have mentioned before, we would like to determine the space-time generated by a spherically symmetric, minimally coupled, self-similar, collapse of a massless scalar field in four-dimensions. Therefore, we shall write our metric ansatz as,

$$ds^2 = -2e^{2\sigma(u,v)}dudv + r^2(u,v)d\Omega^2, \quad (1)$$

where $\sigma(u, v)$ and $r(u, v)$ are two arbitrary functions to be determined by the field equations, (u, v) is a pair of null coordinates varying in the range $(-\infty, \infty)$, and $d\Omega^2$ is the line element of the unit sphere.

The scalar field Φ will be a function only of the two null coordinates and the expression for its stress-energy tensor $T_{\alpha\beta}$ is given by [9],

$$T_{\alpha\beta} = \Phi_{,\alpha}\Phi_{,\beta} - \frac{1}{2}g_{\alpha\beta}\Phi_{,\lambda}\Phi^{,\lambda}. \quad (2)$$

where $,$ denotes partial differentiation.

Now, combining Eqs. (1) and (2) we may obtain the Einstein's equations which in the units of Ref. [9] and after re-scaling the scalar field, so that it absorbs the appropriate numerical factor, take the following form,

$$2(rr_{,uv} + r_{,u}r_{,v}) + e^{2\sigma} = 0, \quad (3)$$

$$2rr_{,vv} - 4rr_{,v}\sigma_{,v} = -r^2(\Phi_{,v})^2, \quad (4)$$

$$2rr_{,uu} - 4rr_{,u}\sigma_{,u} = -r^2(\Phi_{,u})^2, \quad (5)$$

$$2(r^2\sigma_{,uv} + rr_{,uv}) = -r^2(\Phi_{,u}\Phi_{,v}), \quad (6)$$

The equation of motion for the scalar field, in these coordinates, is

$$r\Phi_{,uv} + r_{,u}\Phi_{,v} + r_{,v}\Phi_{,u} = 0. \quad (7)$$

The above system of non-linear, second-order, coupled, partial differential equations (3)-(7) can be solved if we explore the fact that it is continuously self-similar. More precisely, the solution assumes the existence of an homothetic Killing vector of the form,

$$\xi = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \quad (8)$$

Following Coley [10], equation (8) characterizes a self-similarity of the first kind. We can express the solution in terms of the variable $z = v/u$.

In order to obtain our solution, we shall re-write the above equations (3-7) in terms of z , in a slightly different way than previous works [6]. We shall write the unknown functions as: $\Phi(u, v) = \Phi(z)$, $\sigma(u, v) = \sigma(z)$ and $r(u, v) = r(z)$. The system of equations (3-7) will have the mentioned self-similarity if we re-write $e^{2\sigma}$ in (1) in the following way,

$$e^{2\sigma} = -f_{,u}f_{,v}, \quad (9)$$

where $f = f(u, v)$ is an arbitrary function of the null coordinates u and v and we must also write it as: $f(u, v) = f(z)$. It is clear that the main difference between our work and previous ones is in the way we re-write $\sigma(u, v)$ and $r(u, v)$ as functions of z .

In terms of z and taking in account (9), the system of equations (3-7) becomes,

$$2r(\dot{r} + z\ddot{r}) + 2z(\dot{r})^2 - zF^2 = 0, \quad (10)$$

$$2z\ddot{r} - \frac{2\dot{r}}{F}(F + 2z\dot{F}) = -zr(\dot{\Phi})^2, \quad (11)$$

$$2z\ddot{r} + 2\dot{r} + \frac{2r}{F^2}[F\dot{F} + zF\ddot{F} - z(\dot{F})^2] = -zr(\dot{\Phi})^2, \quad (12)$$

$$(zr^2\dot{\Phi})' = 0, \quad (13)$$

where \cdot means differentiation with respect to z , $\dot{f}(z) \equiv F(z)$ and equations (4) and (5) reduce to equation (11).

The above system, equations (10-13), can be greatly simplified if we re-write it in terms of the new variable $2R \equiv \ln z$. The resulting equations are,

$$(r^2)'' = 4e^{4R}F^2, \quad (14)$$

$$2r'' - 4r'\left(2 + \frac{F'}{F}\right) = -r(\Phi')^2, \quad (15)$$

$$2r'' + 2r\left(\frac{F'}{F}\right)' = -r(\Phi')^2, \quad (16)$$

$$(r^2\Phi')' = 0, \quad (17)$$

where the $'$ means differentiation with respect to R .

We solve the system of equations (14-17) by initially writing down a second order differential equation for r in the following way. First of all, we subtract equation (15) from equation (16) and manipulate it in order to find,

$$2 + \frac{F'}{F} = \frac{C}{r^2}, \quad (18)$$

where C is a real integration constant. Then, we differentiate equation (14) once with respect to R and introduce, in the differentiated equation, the value of $e^{4R}F^2$ from equation (14)

and the information coming from equation (18). Finally, we integrate the resulting equation to find,

$$r^3 r'' - 2C r r' = B, \quad (19)$$

where B is a real integration constant. Later, when we show that our solution is indeed the Wyman's one, we shall identify the physical meaning of C and B .

The equation for Φ can be obtained if we eliminate F and its derivatives from either one of the equations (15) or (16), with the aid of equation (18). They give the same result, which is,

$$\Phi' = \frac{\sqrt{-2B}}{r^2}. \quad (20)$$

It is easy to see that equation (17) also leads to the same result above. Another way to write Φ , which will be very important to our work, is derived from equation (20) with the aid of equation (18). After some algebra and an integration, we get,

$$\Phi = \frac{\sqrt{-2B}}{2C} \ln(2e^{4R} F^2), \quad (21)$$

where we set the integration constant to $(1/2) \ln 2$.

Therefore, a solution to the system of equations (14-17) is computed by initially solving equation (19). Then, for the function $r(R)$ obtained, one uses equation (14) to find $e^{2\sigma}$. Finally, for the function $r(R)$ obtained, one uses equation (20) to find the scalar field $\Phi(R)$ or equivalently for the value of $e^{4R} F^2$ obtained, one uses equation (21) to find $\Phi(R)$.

Now, we would like to demonstrate that the Wyman's solution is a solution to the system of equations (14-17). As the first step in our demonstration, we apply the following coordinate transformations,

$$U = \ln u, \quad V = \ln v. \quad (22)$$

From the definition of $2R$ as $\ln z$, we immediately obtain from equation (22) that, $2R = V - U$. In terms of these new coordinates, the line element equation (1) becomes,

$$ds^2 = -2e^{4R}F(R)^2dVdU + r(R)^2d\Omega^2. \quad (23)$$

We may now compare it with the line element of the Wyman's solution written in the double null coordinates (U,V) [3],

$$ds^2 = - \left(1 - \frac{2\eta}{\mathcal{R}}\right)^{M/\eta} dVdU + \left(1 - \frac{2\eta}{\mathcal{R}}\right)^{1-M/\eta} \mathcal{R}^2 d\Omega^2, \quad (24)$$

where M is the mass parameter, $\eta^2 = M^2 + \Sigma^2$ and Σ is the scalar charge. The function $\mathcal{R}(U, V)$ may be written in terms of R in the following way,

$$R - R_0 = \frac{1}{1 + M/\eta} \left[\left(1 - \frac{2\eta}{\mathcal{R}}\right)^{-M/\eta} \mathcal{R} \left(1 - \frac{\mathcal{R}}{2\eta}\right)^{M/\eta} {}_2F_1\left(1 + \frac{M}{\eta}, \frac{M}{\eta}; 2 + \frac{M}{\eta}; \frac{\mathcal{R}}{2\eta}\right) \right] \quad (25)$$

where ${}_2F_1(a, b; c; z)$ is the Gauss hypergeometric function and R_0 is an integration constant. This is a transcendental equation which cannot be inverted, in general, to give us \mathcal{R} as an explicit function of R . Nevertheless, we shall consider \mathcal{R} as an implicit function of R and use equation (25) to prove our thesis.

Now, since r and \mathcal{R} are functions of R , we may directly compare the metric components in the line elements (23) and (24) to obtain,

$$r(R)^2 = \left(1 - \frac{2\eta}{\mathcal{R}(R)}\right)^{1-M/\eta} \mathcal{R}(R)^2 \quad (26)$$

and

$$2e^{4R}F(R)^2 = \left(1 - \frac{2\eta}{\mathcal{R}(R)}\right)^{M/\eta} \quad (27)$$

Therefore, from the Wyman's solution we have an expression for r , which must satisfy equation (19) and an expression for $e^{2\sigma} = e^{4R}F(R)^2$, which must satisfy equation (14).

Introducing the expression of r in terms of \mathcal{R} (26) in equation (19), we may prove, with the aid of equation (25), that it satisfies equation (19) if, and only if, $C = M$ and $-B = \Sigma^2$. From now on, we shall assume that those are the correct values of the integration constants C and B . Likewise, if we introduce the expressions of r equation (26) and $e^{2\sigma}$ equation (27) in equation (14), we may prove, with the aid of equation (25), that this equation is satisfied.

We complete our demonstration by using the value of $e^{4R}F(R)^2$ equation (27), in equation (21) to find,

$$\Phi = \frac{\Sigma}{2\eta} \ln \left(1 - \frac{2\eta}{\mathcal{R}} \right). \quad (28)$$

Which is exactly the expression for the scalar field in the Wyman's solution [3].

So, the Wyman's solution may be derived from the Einstein-scalar equations (3-6) by using the continuous self-similarity of those equations in an appropriated way. As a matter of completeness we mention that, if we introduce a time coordinate T defined by, $2T = V + U$, we may re-write the line element equation (1), as,

$$ds^2 = 2e^{4R}F(R)^2(-dT^2 + dR^2) + r^2(R)d\Omega^2. \quad (29)$$

III. WYMAN'S SOLUTION AND CRITICAL BEHAVIOUR.

Depending on the value of the parameters M and Σ the Wyman's solution may represent different static, asymptotically flat space-times [1], [3]. When one sets $\Sigma = 0$, the scalar field vanishes from equation (28) and one obtains the Schwarzschild solution with a mass M . Therefore, it is usual to consider M positive. For positive M one may let Σ^2 take values over the real line. For $\Sigma^2 > 0$, the solution represents space-times with an eternal naked time-like singularity located at $r = 0$. From equation (28), the scalar field vanishes asymptotically ($r \rightarrow \infty$) and diverges at the singularity. For $\Sigma^2 < 0$, one normally considers the domain: $-M^2 \leq \Sigma^2 < 0$ because if $\Sigma^2 < -M^2$ from equation (24) the metric gets complex. The case $\Sigma^2 = -M^2$ is well known in the literature as the Yilmaz-Rosen space-time [12]. Observe that this restriction in the domain of Σ^2 , when this quantity is negative, comes from the use of the quantities (U, V, \mathcal{R}) equation (24). When one considers the metric written in the original quantities in the Wyman's papers [2] or even in the quantities leading to equation (23), one sees that the domain of $\Sigma^2 < 0$ may be extended to: $(0, -\infty)$. In the present situation, since we are using the identifications (26) and (27), we can see that our original

quantities (U, V, r) become ill defined for $\Sigma^2 < -M^2$. Therefore, we shall consider here the domain of Σ^2 to be $[-M^2, \infty)$. In these space-times, r is never zero. If one starts with a large value of r it diminishes, as we let it varies as a function of R , until it reaches a minimum value. Then, it starts to increase again without limit. We may interpret this case as an eternal bouncing solution. An important property of this space-time is that the scalar field equation (28) is imaginary. The imaginary scalar field also known as ghost Klein-Gordon field [13] is an example of the type of matter called *exotic* by some authors [14]. It violates most of the energy conditions and is repulsive. This property helps explaining the reason why the collapsing scalar field bounces back without reaching $r = 0$. Recently, the exotic matter has been in evidence due to the discovery that the universe is expanding in an accelerated rate [15]. This implies that the universe must be filled with matter which violates at least the strong energy condition [15]. This type of matter is also very much used as the matter responsible for the formation and maintenance of traversible wormholes [14], [16]. In particular, the ghost Klein-Gordon field has been used as one of the first specific *exotic* matter models to explain the formation and maintenance of traversible wormholes [13]. Besides the above motivations for the use of *exotic* matter, which involve classical fields, one must not forget about negative energy densities very common in quantum field theories [17].

Based on the above properties of the Wyman's solution and taking in account the results of references [6], [7], [8], on critical behaviour in the spherical symmetric, massless, scalar field collapse, we conclude that the Wyman's solution also shows a critical behaviour. Although the critical solution is of a different type from the one discovered in the above mentioned references, one may proceed here in the same way as the authors there, in order to show the critical behaviour. One fixes the mass parameter M and considers the scalar charge Σ as the only free parameter. In the present case, as discussed above, we fix M to a positive value, say M_0 . Σ^2 may take values in the domain, $(-M_0^2, \infty)$. For Σ^2 positive ($0 < \Sigma^2 < \infty$), we have a set (N) of solutions representing asymptotically flat, time-like, eternal naked singularities. Consider, now, equation (19), suitably written for the present situation as,

$$r^3 r'' - 2M_0 r r' = -\Sigma^2. \quad (30)$$

We may numerically solve it using boundary conditions reflecting the asymptotic behaviour of the solution and the appropriate value of Σ^2 . With this numerical solution, we may draw the curve $r \times R$. Following the above procedure, we show in Figure 1 the curve $r \times R$ for two members of the set N . For Σ^2 negative ($-M_0^2 < \Sigma^2 < 0$), we have a set (Bo) of solutions representing asymptotically flat, eternal bouncing space-times. Figure 2 shows the curve $r \times R$ for two members of the set Bo , obtained from the numerical solutions of equation (30) with appropriate values of Σ^2 . Finally, for $\Sigma^2 = 0$, we obtain the critical solution which interpolates between the infinity number of solutions in the set N and the ones in the set Bo . For fix M this solution is a point in the parameter space of solutions and represents the Schwarzschild black hole with mass M_0 . We may solve equation (30) for the appropriate value of Σ^2 and draw the curve $r \times R$ for the Schwarzschild black hole. This curve is shown in Figure 3.

FIGURES

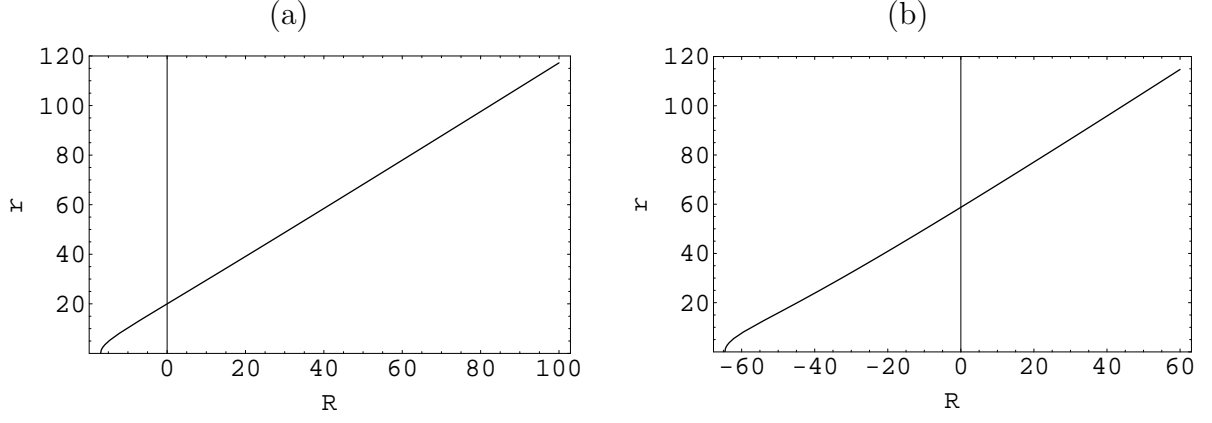


FIG. 1. Graphics of $r \times R$ for the eternal naked singularities with different values of M_0 and Σ^2 .

(a) $M_0 = 1$ and $\Sigma^2 = 160$, (b) $M_0 = 3$ and $\Sigma^2 = 280$.

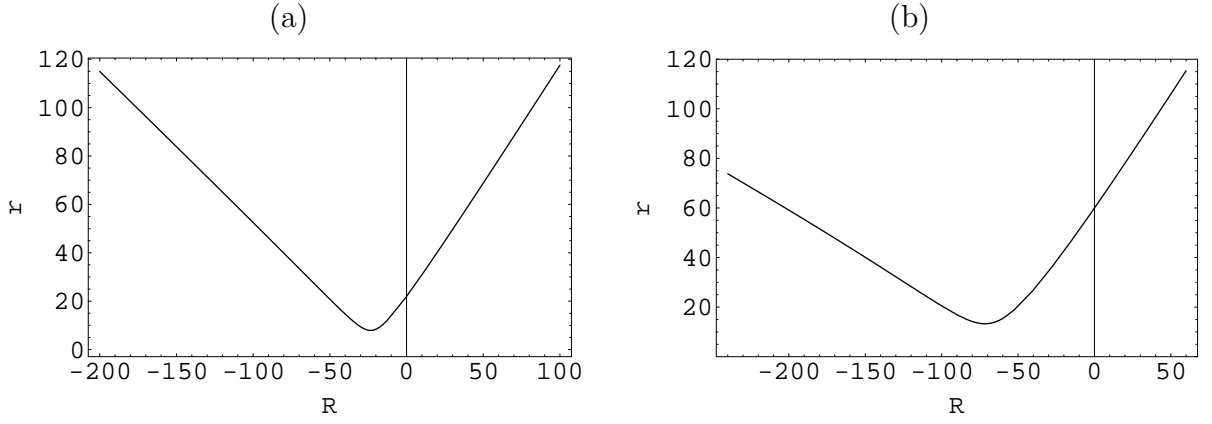


FIG. 2. Graphics of $r \times R$ for the eternal bouncing space-times with different values of M_0 and Σ^2 . (a) $M_0 = 1$ and $\Sigma^2 = -160$, (b) $M_0 = 3$ and $\Sigma^2 = -280$.

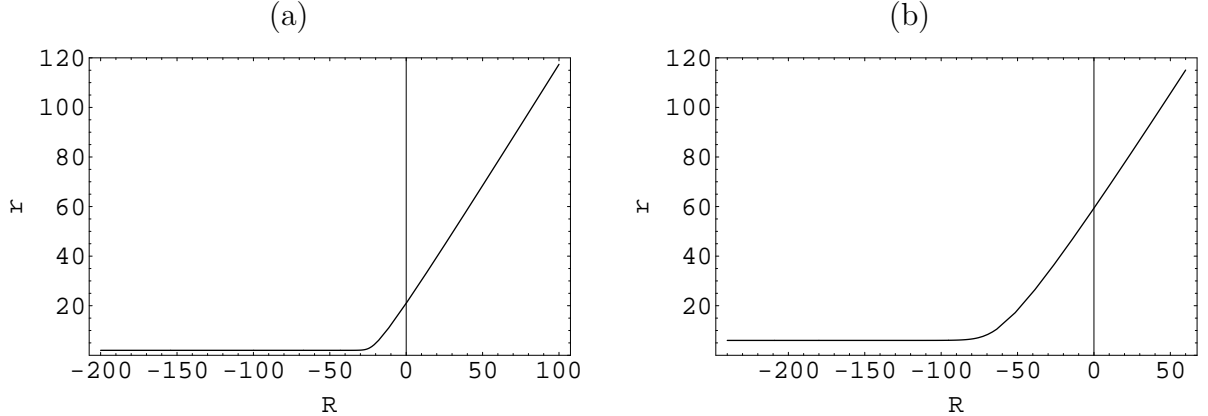


FIG. 3. Graphics of $r \times R$ for the Schwarzschild space-times with different values of M_0 . (a) $M_0 = 1$ (b) $M_0 = 3$.

IV. CONCLUSIONS.

In the present work, we showed that the Wyman's solution may be obtained from the four-dimensional Einstein's equations for a spherically symmetric, minimally coupled, massless scalar field by using the continuous self-similarity of those equations. We also showed that the Wyman's solution exhibits critical behaviour. We did that by fixing the mass M to a positive value, say M_0 , and letting Σ^2 takes values along the real line. For $\Sigma^2 > 0$ the space-times have eternal naked singularities, for $\Sigma^2 = 0$ one has a Schwarzschild black hole of mass M_0 and finally for $-M_0^2 \leq \Sigma^2 < 0$ one has eternal bouncing solutions. Here, the critical solution is a Schwarzschild black hole which interpolates between an infinity number of eternal naked singularities and bouncing space-times. Although, this behaviour is very different from the one discovered by Choptuik, one may have other types of critical solutions in the Einstein-scalar system, as demonstrated by P. Brady in [11].

ACKNOWLEDGMENTS

We would like to thank FAPEMIG for the invaluable financial support.

REFERENCES

- [1] O. Bergman and R. Leipnik, Phys. Rev. D **107**, 1157 (1957); H. A. Buchdahl, *ibid.* bf 111, 1417 (1959); A. I. Janis, E. T. Newman and J. Winicour, Phys. Rev. Lett. **20**, 878 (1968); J. E. Chase, Commun. Math. Phys. bf 19, 276 (1970); A. G. Agnese and M. LaCamera, Lett. Nuovo Cimento **35**, 365 (1982); Phys. Rev. D **31**, 1280 (1985); D. D. Dionysion, Astro. Space Sci. **83**, 493 (1982); J. Froyland, Phys. Rev. D. **25**, 1470 (1982); M. D. Roberts, Gen. Rel. Grav. **17**, 913 (1985); Class. Quant. Grav. **2**, L69 (1985); Astro. Space Sci. **147**, 321 (1988).
- [2] M. Wyman, Phys. Rev. D **24**, 839 (1981).
- [3] M. D. Roberts, Gen. Rel. Grav. **21**, 907 (1989).
- [4] P. Jetzer and D. Scialom, Phys. Lett. A **169**, 12 (1992).
- [5] M. A. Clayton, L. Demopoulos and J. Legare, Phys. Lett. A **248**, 131 (1998).
- [6] P. R. Brady, Class. Quantum Grav. **11**, 1255 (1994); Y. Oshiro, K. Nakamura and A. Tomimatsu, Prog. Theor. Phys. **91**, 1265 (1994).
- [7] M. W. Choptuik, Phys. Rev. Lett. **70**, 9 (1993).
- [8] D. Christodoulou, Commun. Math. Phys. **105**, 337 (1986); **106**, 587 (1986); **109**, 591 and 613 (1987); Commun. Pure Appl. Math. **XLIV** 339 (1991); **XLVI**, 1131 (1993); Ann. Math. **140**, 607 (1994).
- [9] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, (Freeman, New York, 1973).
- [10] A. A. Coley, Class. Quantum Grav. **14**, 87 (1997).
- [11] P. R. Brady, Phys. Rev. D **51**, 4168 (1995).
- [12] H. Yilmaz, Phys. Rev. **111**, 1417 (1958); N. Rosen, in Relativity, eds. Carmeli, Fickler and Witten (New York, N. Y., 1970), 229.

- [13] S. A. Hayward, S. W. Kim and H. Lee, Phys. Rev. D **65**, 064003 (2002).
- [14] M. S. Morris and K. S. Thorne, Am. J. Phys. **56**, 395 (1988).
- [15] For a review of astrophysical observations and physical models leading to these conclusions see: N. Straumann, in Proceedings of the First Séminaire Poincaré, (Paris, France, 2002); astro-ph/0203330.
- [16] For a review of recent work in this area see: S. A. Hayward, in Proceedings of the 12th Workshop on General Relativity, (Tokyo, Japan, 2002); gr-qc/0306051.
- [17] N. D. Birrell and P. C. W. Davies, Quantum Fields in Curved Space, (Cambridge University Press, Cambridge, 1982).